# Generalized Linear Models

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# Review and Introduction

- Let $y_1, ..., y_n$ denote $n$ independent observations on a response.
- Treat $y_i$ as a realization of a random variable $Y_i$
- In the general linear model we assume that

$$Y_i \sim N(\mu_i, \sigma^2)$$

- And we further assume that the expected value $\mu_i$ is a linear function

$$\mu_i = X_i' \beta$$

- The generalized linear model generalizes both the random and systematic component.

# Components of Generalized Linear Models

- All generalized linear models have three components:
  - Random component
  - Systematic component
  - Link function

# Random Component

- The random component of a GLM identifies the response variable Y and selects a probability distribution for it.
- Denote the observations on $Y$ by $(Y_1, Y_2, ..., Y_n)$. Standard GLMs treat $Y_1, Y_2, ..., Y_n$ as independent.
- If the observations on $Y$ are binary then we assume a *binomial distribution* for $Y$
- In some applications, each observation is a count. Then we have *Poisson or Negative Binomial*
- If each observation is continuous, we might assume a normal distribution for Y.

# Systematic Component

- The systematic component of a GLM specifies the explanatory variables.
- These enter linearly as predictors on the right-hand side of the model equation.
- The systematic component specifies the variables that are the $\{x_j\}$ in the formula

$$\alpha + \beta_1 x_1 + ... + \beta_k x_k$$

# Link Function

- Denote the expected value of $Y$ the mean of the probability distribution by $\mu = E(Y)$
- The link function specifies a function $g(.)$ that relates $\mu$ to the linear predictors as

$$g(\mu) = \alpha + \beta_1 x_1 + \ldots + \beta_k x_k$$

- The function $g(\mu)$ the link function connects the random and the systematic components.

# The exponential Family

- We assume that observations come from a distribution in the exponential family with the following probability density function:

$$f(y_i;\theta_i,\phi) = exp\left\{\frac{y_i\theta_i}{a(\phi)} + c(y_i,\phi)\right\} \tag{1}$$

- Here $\theta_i, \phi$ are parameters and $a(.), b(.)$ and $c(.)$ are known functions.

- The $\theta_i$ and $\phi$ are location and scale parameters respectively.

# Example 1: Normal Distribution

- The normal distribution is given as:

$$f(y_i, \theta_i, \phi) = \frac{1}{\sqrt{2\pi\sigma}} exp\{-\frac{1}{2\sigma^2}(y_i - \mu)^2\}$$

- Which can be expressed as:

$$f(y_i, \theta_i, \phi) = \exp\left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y_i^2 - 2y_i\mu + \mu^2)\right]$$

- We can re-factor and have:

$$f(y_i, \theta_i, \phi) = \left(\frac{2\mu y_i - \mu^2}{2\sigma^2}\right) - \frac{1}{2}\left(\frac{y_i^2}{\sigma^2} + \log(2\pi\sigma^2)\right)$$

# Solution Cont'd

- $\theta_i = \mu, \phi = \sigma^2, a_i(\phi) = \phi, b(\theta_i) = \frac{\theta_i^2}{2}, c(y_i, \phi) = \frac{1}{2}\left(\frac{y_i^2}{\sigma^2} + \log(2\pi\sigma^2)\right)$
- The mean is given as $E(y_i) = b'(\theta_i)$
- The variance $Var(y_i) = b''(\theta_i)a(\phi)$

# Exercise 1: Poisson distribution

- The PMF of the Poisson distribution is given as:

$$f(y|\mu) = \frac{e^{-\mu}\mu^{y}}{y!}$$

- Show that the Poisson Distribution can be expressed as a member of exponential family and derive the mean and variance.

# Exercise 2: Binomial distribution

- The PMF of the Binomial distribution is given as:

$$f(y|n,p) = \binom{n}{y} p^{y}(1-p)^{n-y}$$

- Show that the binomial Distribution can be expressed as a member of exponential family and derive the mean and variance.

# Exercise 3

- The PMF of the Negative Binomial distribution is given as:

$$f(y|r,p) = \binom{r+y-1}{y} p^r (1-p)^y$$

- Show that the negative binomial Distribution can be expressed as a member of exponential family and derive the mean and variance.

# Maximum Likelihood Estimation of GLM

- Unlike for the general linear model, there is no closed form expression for the MLE of $\beta$ in general for GLMs.
- However all the GLMs can be fit using the same algorithm a form of iteratively re-weighted least squares
- Given an initial value for $\hat{\beta}$ calculate the estimated linear predictor $\hat{\eta}_i = x_i'\beta$ and use that to obtain the fitted values $\hat{\mu}_i = g^{-1}(\hat{\eta}_i)$. Calculate the adjusted dependent variable

$$z_i = \hat{\eta}_i + (y_i - \hat{\mu}_i)(\frac{d\eta_i}{d\mu_i})_0$$

# Cont'd

- Calculate the iterative weights

$$W_i^{-1} = (\frac{d\eta_i}{d\mu_i})V_i$$

  where $V_i$ is the variance function evaluated at $\hat{\mu}_i$

- Regress $z_i$ on $x_i$ with weight $W_i$ to give the new estimate of $\beta$

# Thank You!